# Correspondence

## Verification of the Westervelt Equation for Focused Transducers

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***Abstract*—This study investigates the validity of the Westervelt equation for focused transducers. The angular spectrum method is employed to analyze the second-harmonic acoustic field under the weakly nonlinear approximation. Although it is well known that the Westervelt equation is accurate for the case of quasi-plane waves, the present work demonstrates accurate solution for the highly focused case of a spherically-curved ultrasound transducer having an aperture angle of 80°. It is further found that the solution error is inversely dependent on the nonlinearity coefficient.**

### I. Introduction

Simulation of focused sources has been carried out primarily using the Kuznetsov-Zabolotskaya-Khokhlov (KZK) equation [1]–[3] and the Westervelt equation [4]–[6]. The KZK equation represents the parabolic approximation to the Westervelt equation, and thus is only valid for solutions distant from the source and composed of propagation angles not deviating far from the axis of propagation [7]. The Westervelt equation provides a more complete description, but also holds approximations: it is valid when local effects can be ignored, making it reasonably accurate for progressive quasi-plane waves (weakly focusing transducers) at distances greater than one wavelength away from the source [7]. The present study considers the accuracy of the Westervelt equation under the condition of a focusing field.

Kamakura *et. al* [8] previously addressed this topic for a study of aperture angles up to 30° in the appendix, noting that "further investigation is essential to evaluate nonlinearity source terms adequately." Tsuchiya [9] presented a related study on the comparison between Kuznetsov's equation and the parabolic approximation, but did not include the Westervelt equation.

A systematic investigation of error under focusing conditions is presented, extending upon Kamakura's work by considering aperture angles of up to 80°. To allow a more precise comparison between Kuznetsov's equation and the Westervelt equation, the nonlinear source term is treated in a more exact manner than previously reported [8].

Several methods have been reported for solving the Westervelt equation [5], [6], [10]. To assure correct implementation, however, it is crucial to know when the equation is a good approximation to a given problem. This can be especially important for modeling fields containing a range of wave-vector directions, e.g., fields from focused sources that are used in therapeutic ultrasound [11]. This paper concentrates on comparing the solution of the Westervelt equation with the full finite-amplitude wave equation for focused sources. From this comparison, the validity of the Westervelt equation under highly focused conditions is assessed.

Specifically, a weakly nonlinear problem is considered, because analytic solutions exist in this case [12] for both the Westervelt equation and the full finite-amplitude wave equation. Furthermore, these two solutions differ only in the effective nonlinearity coefficient, as will be shown. Because of this similarity, the same algorithm has been used to calculate both solutions. In this way algorithm calculation differences, which may contribute to overall solution differences, can be ruled out. This eliminates a common problem in comparison studies, in which it is often difficult, if not impossible, to tell if differences are caused by the equations being compared or by differences in the algorithms themselves.

### II. Theory

For an ideal fluid, nonlinear wave propagation can be described by the following second-order wave equation [12]–[14]:

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p(\mathbf{r},t) + \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}p^2(\mathbf{r},t) + \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)L = 0, \tag{1}$$

where $L = 1/2(\rho_0 \boldsymbol{v}\cdot\boldsymbol{v} - p^2/\rho_0 c_0^2)$ is the Lagrangian density of acoustical energy, $p$ is the sound pressure, $c_0$ is the sound speed, $\beta$ is the nonlinearity coefficient, $\rho_0$ is the ambient density, and $\boldsymbol{v}$ is the velocity vector. Here, attenuation has been neglected, but can be added *ad hoc*.

Eq. (1) can be simplified to the Westervelt equation for directional beams by discarding the $L$ term, and is written as [4]

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p(\boldsymbol{r},t) + \frac{\beta}{\rho_0 c_0^4}\frac{\partial^2}{\partial t^2}p^2(\boldsymbol{r},t) = 0. \tag{2}$$

These equations will be solved by the angular spectrum approach [12], [15]. Considering a continuous-wave, weakly nonlinear case, in which the pressure at the fundamental frequency is denoted $P_1(\boldsymbol{r})$, and the pressure at

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the second harmonic is denoted $P_2(\boldsymbol{r})$, it was found that a partial differential equation for $P_2$ can be written as [12]

$$(\nabla^2 + 4k^2)P_2 = \frac{2k^2}{\rho_0 c_0^2}\int\int \tilde{\beta}(\boldsymbol{k}', \boldsymbol{k}'')e^{i(\boldsymbol{k}'+\boldsymbol{k}'')\cdot \boldsymbol{r}} \times \tilde{P}_1(\boldsymbol{\kappa}', 0)\tilde{P}_1(\boldsymbol{\kappa}'', 0)\frac{\mathrm{d}\boldsymbol{\kappa}'\mathrm{d}\boldsymbol{\kappa}''}{(2\pi)^4}, \tag{3}$$

where $k$ is the wave number at the fundamental frequency, $\tilde{P}_1$ is the pressure at the fundamental frequency in the k-space. In terms of position vector $r(\boldsymbol{x}, z)$, it is assumed that the initial projection plane is at $z = 0$. Furthermore, for the $n$th harmonics,

$$K_n = \sqrt{(nk)^2 - |\boldsymbol{\kappa}|^2}, \quad n = 1, 2, \tag{4}$$

$\boldsymbol{k} = (\boldsymbol{\kappa}, K_1)$, $\boldsymbol{r} = (\boldsymbol{x}, z)$. Finally, if $\theta$ is the angle between direction $\boldsymbol{k}'$ and $\boldsymbol{k}''$, we have [12], [14], [16]

$$\tilde{\beta} = B/2A + \cos^4(\theta/2) = \beta - (1 - \cos^4(\theta/2)) \tag{5}$$

and $\boldsymbol{k}' \cdot \boldsymbol{k}'' = k^2\cos(\theta)$. Fourier transformation with respect to $x$ on both sides of (3) yields

$$\left(\frac{\partial^2}{\partial z^2} + K_2^2\right)\tilde{P}_2 = \frac{2k^2}{\rho_0 c_0}\int \tilde{\beta}(\boldsymbol{k}', \boldsymbol{k} - \boldsymbol{k}')e^{i(K_a + K_b)z} \times \tilde{P}_1(\boldsymbol{\kappa}', 0)\tilde{P}_1(\boldsymbol{\kappa} - \boldsymbol{\kappa}', 0)\frac{\mathrm{d}\boldsymbol{\kappa}'}{(2\pi)^2}, \tag{6}$$

where $\tilde{P}_2$ is the pressure at the second harmonic in the k-space, and

$$K_a = \sqrt{k^2 - |\boldsymbol{\kappa}'|^2}, \quad K_b = \sqrt{k^2 - |\boldsymbol{\kappa} - \boldsymbol{\kappa}'|^2}. \tag{7}$$

For a weakly focusing (directional) transducer, for which $\tilde{\beta}$ can be replaced by $\beta$, (6) can be reduced to

$$\left(\frac{\partial^2}{\partial z^2} + K_2^2\right)\tilde{P}_2 = \frac{2\beta k^2}{\rho_0 c_0}\int e^{i(K_a + K_b)z}\tilde{P}_1(\boldsymbol{\kappa}', 0)\tilde{P}_1(\boldsymbol{\kappa} - \boldsymbol{\kappa}', 0)\frac{\mathrm{d}\boldsymbol{\kappa}'}{(2\pi)^2}, \tag{8}$$

which is [12, Eq. (37)]. The analytic solution to (6) can be derived [17] and is written as

$$\tilde{P}_2 = \int Q_f(\boldsymbol{\kappa}, \boldsymbol{\kappa}', z)\tilde{\beta}(\boldsymbol{k}', \boldsymbol{k} - \boldsymbol{k}')\tilde{P}_1(\boldsymbol{\kappa}', 0)\tilde{P}_1(\boldsymbol{\kappa} - \boldsymbol{\kappa}', 0)\frac{\mathrm{d}\boldsymbol{\kappa}'}{(2\pi)^2}, \tag{9}$$

where

$$Q_f(\boldsymbol{\kappa}, \boldsymbol{\kappa}', z) = \frac{2k^2}{\rho_0 c_0}\left(\frac{e^{i(K_a + K_b)z} - e^{iK_2 z}}{K_2^2 - (K_a + K_b)^2}\right). \tag{10}$$

The pressure of the second harmonic can be finally calculated by applying the inverse Fourier transform to $\tilde{P}_2$.

Similarly, a solution to (8) can be derived, which is equal to [12, Eq. (39)]. Subsequently, (9) is denoted the full solution [the solution of (1)] whereas [12, Eq. (39)] is denoted the partial solution [the solution of (2), i.e., the Westervelt equation]. It has been argued that replacing $\beta$ with $\tilde{\beta}$ is equivalent to neglecting the term $L$ in (1) [7], [12], which results in the Westervelt equation. The goal is now to numerically compare the complete solution and the partial solution for focused transducers with different angles.

## III. Simulation

The transducer under test had a radius of 20 mm. Noting that 20° represents the approximate angular validity limit of the KZK equation [8], the aperture angle was varied from 20° to 80° at increments of 20°. The excitation signal was continuous wave at 1 MHz, and had an amplitude of 100 kPa. The acoustic medium was water, and assumed to have a sound speed of 1500 m/s, density 1000 kg/m$^3$, and nonlinear coefficient of 3.5. Attenuation was neglected. The focal gain was defined as $G = ka^2/2d$, where $d$ is the focal length and $a$ is the radius. It was varied from 14 to 41 in this study.

For the numerical simulation, the area of the plane was chosen to be large enough to prevent artificial wraparound. The spatial step-size was a quarter wavelength of the fundamental frequency. To consider the focusing effect, a phase function $e^{ik(R-d)}$ was applied to the pressure distribution on the transducer, where $R$ is the distance from the focal point to a point on the transducer plane [7].

Fig. 1 shows the second-harmonics sound pressure distribution along the axis at four different aperture angles. The distance was normalized by the focal length and the pressure amplitude was also normalized. Error, which was defined as $(p_{\text{partial}} - p_{\text{complete}})/p_{\text{complete}}$, can be found in the plots as well. The least square error was also used, and was defined by

$$\text{error} = \frac{\| p_{\text{partial}}(z) - p_{\text{complete}}(z) \|}{\| p_{\text{complete}}(z) \|}. \tag{11}$$

It can be seen from Fig. 1 that, for both large and small aperture angles, the complete and partial solutions give almost identical values. Their pressure distributions along the axis are visually indistinguishable. Most error values in the plot are below 5%. The biggest discrepancy for the aperture angle 80° is about 0.8 dB. The least square errors are 0.0276, 0.0239, 0.0165, and 0.0066, for aperture angles 80°, 60°, 40°, and 20°, respectively. As expected, the error decreases with decreasing aperture angle, i.e., the Westervelt equation becomes more accurate. Although it might be difficult to see from the plot, the partial solution usually overestimates the sound pressure around the focal point. This finding may be considered reasonable because the method employs an angle-independent nonlinearity

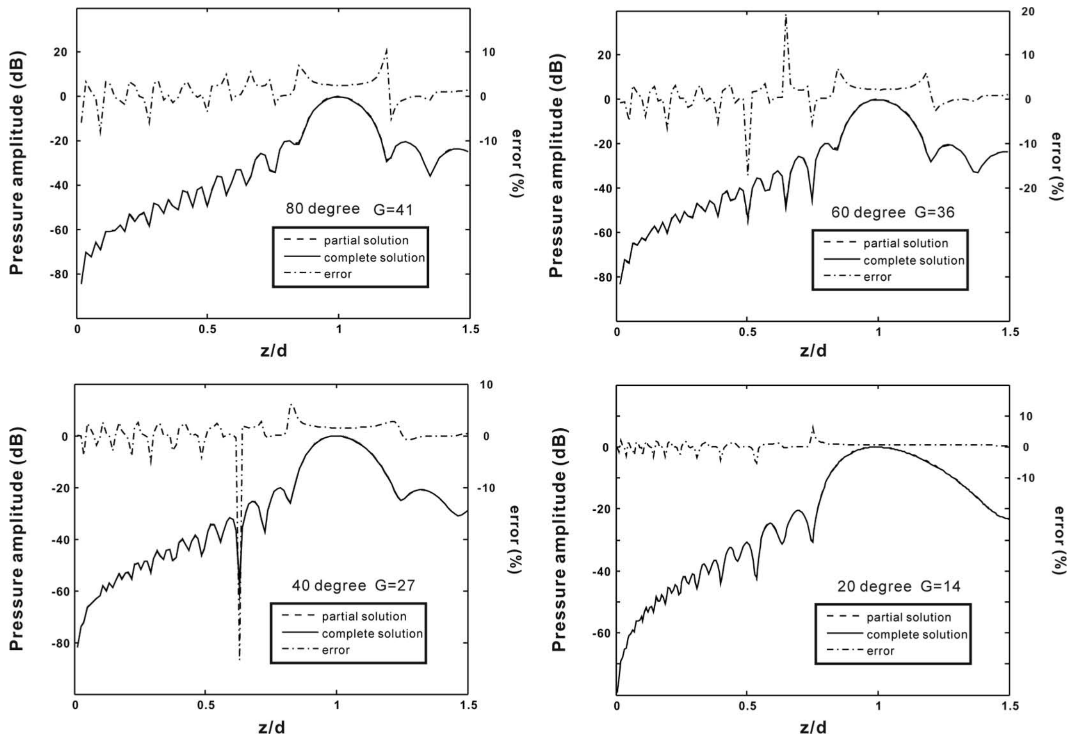


Fig. 1. Comparison of on-axis second harmonics sound pressure distribution between the complete and partial solutions for different aperture angles and a nonlinearity coefficient of 3.5. Errors are also presented.

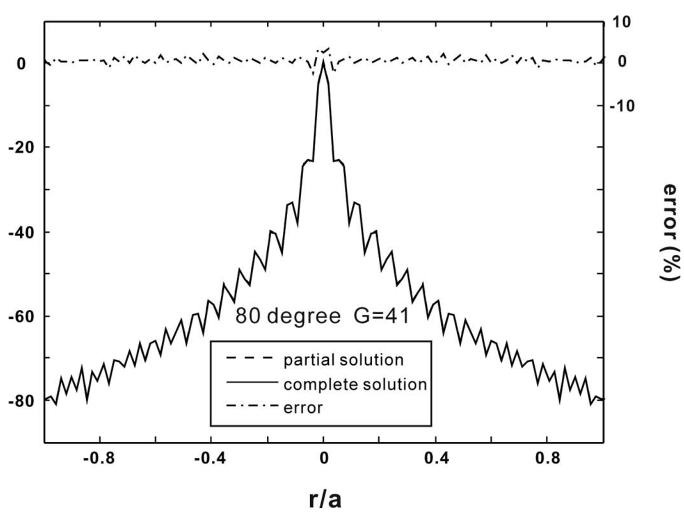


Fig. 2. Comparison of second harmonics sound pressure distribution along the radial direction between the complete and partial solutions at an aperture angle of 80° and a nonlinearity coefficient of 3.5.

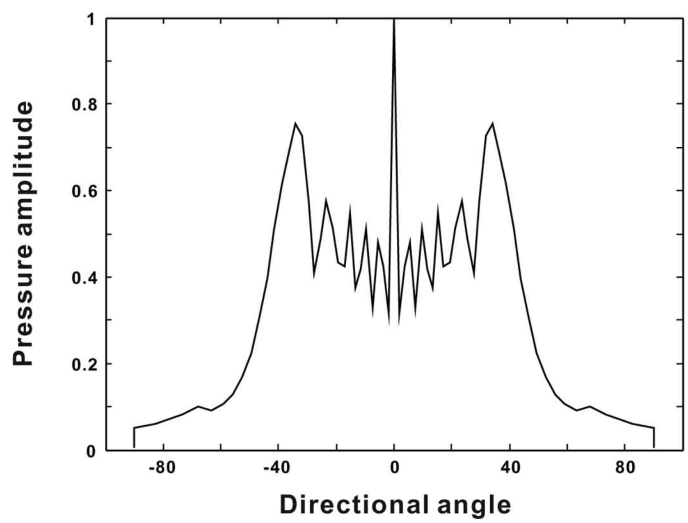


Fig. 3. Pressure distribution in the k-space on the transducer plane, where $k_x$ was varied while $k_y = 0$.

coefficient $\beta$, whereas the complete solution uses an angle-dependent nonlinearity coefficient that is smaller than $\beta$, especially at large $\theta$. Fig. 2 shows the pressure distribution along the radial length across the focal plane for the aperture angle of 80° only. Again, the partial solution does not deviate significantly from the complete solution, and the least square error is 0.0278. Additional simulations show that the same conclusion holds at other distances to the source plane. It is noted that, in [8], the relevant equation of the Lagrangian density is derived based on the assumption that the axial component $u$ in the particle velocity is larger than the radial component $v$ and that the derivative $\partial u/\partial z$ dominates over $\partial u/\partial r$ in the whole space, where $z$ is the axial direction and $r$ is the radial direction. These assumptions are justified only at a location very close to the axis and the field is treated as a Gaussian beam [8]. In contrast, the present study does not make these assumptions, which allows the study of the sound field along the radial direction.

Here, we briefly examine why only small differences have been observed between the complete solution and the partial solution. Fig. 3 shows the pressure distribution in k-space on the transducer plane, where only $k_x$ was varied while $k_y$ was fixed at 0. It can be seen that most of the energy is still within the range of small angles, even though the transducer is strongly focused. The pressure drops drastically when the angle goes beyond approximately $\pm 40°$. Therefore, in the worst case scenario, two plane waves directed at $+40^{\circ}$ and $-40^{\circ}$ result in an effective nonlinearity coefficient of 2.84, which is a 20% decrease from the angle-independent nonlinearity coefficient, i.e., 3.5 in this case. For a plane wave traveling normal to the plane (this direction has the strongest energy) and a plane wave traveling at $\pm 35°$ (this direction has the second strongest energy), the effective nonlinearity coefficient is 3.3, nearly the same as the angle independent nonlinearity coefficient (a 5% decrease). Considering that a significant amount of energy is contained within the angles smaller than $\pm 35°$, the error introduced by the partial solution should be close to 5%, which agrees well with the simulation results.

Now, considering wave propagation in a medium with a small nonlinearity coefficient (e.g., 1.2 as in air), two plane waves traveling at $+40^{\circ}$ and $-40^{\circ}$ would result in an effective nonlinearity coefficient of 0.54—more than a 50% decrease. Similarly, for a plane wave traveling normal to the plane and a wave traveling at $\pm 35°$, the effective nonlinearity coefficient will be reduced by almost 15%. As a result, the error introduced by the partial solution for small nonlinearity coefficient is expected to be larger. A similar conclusion has also been found in [9]. Fig. 4 shows the results for an aperture angle $80^{\circ}$, but with the nonlinearity coefficient changed to 1.2. The biggest discrepancy is about 2.5 dB. The least square error is 0.0843.

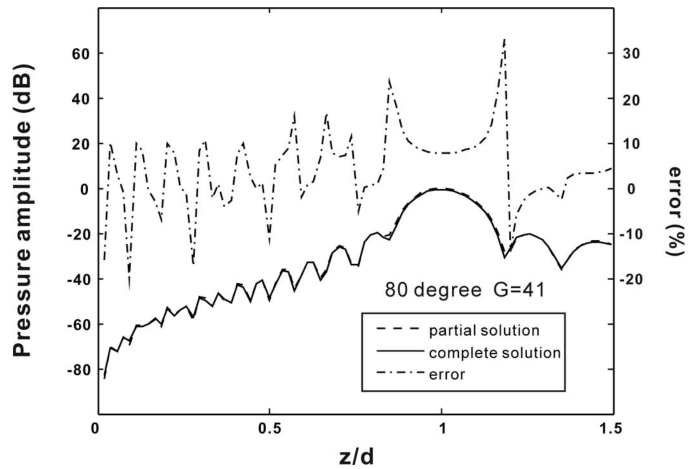


Fig. 4. Comparison of on-axis second-harmonics sound pressure distribution between the complete and partial solutions for an aperture angle 80° and a nonlinearity coefficient of 1.2. Errors are also presented.

Finally, it is also interesting to study the impact of the value of $ka$ on the validity of the Westervelt equation. For a large $ka$, as in the cases studied above ($ka = 84$), it can be assumed that the wave is locally plane, i.e., in the high-frequency limit or geometrical acoustics regime, where the Westervelt equation is typically considered valid. For a small $ka$, the Westervelt equation may not be as accurate, even though the case is less practical in therapeutic ultra-

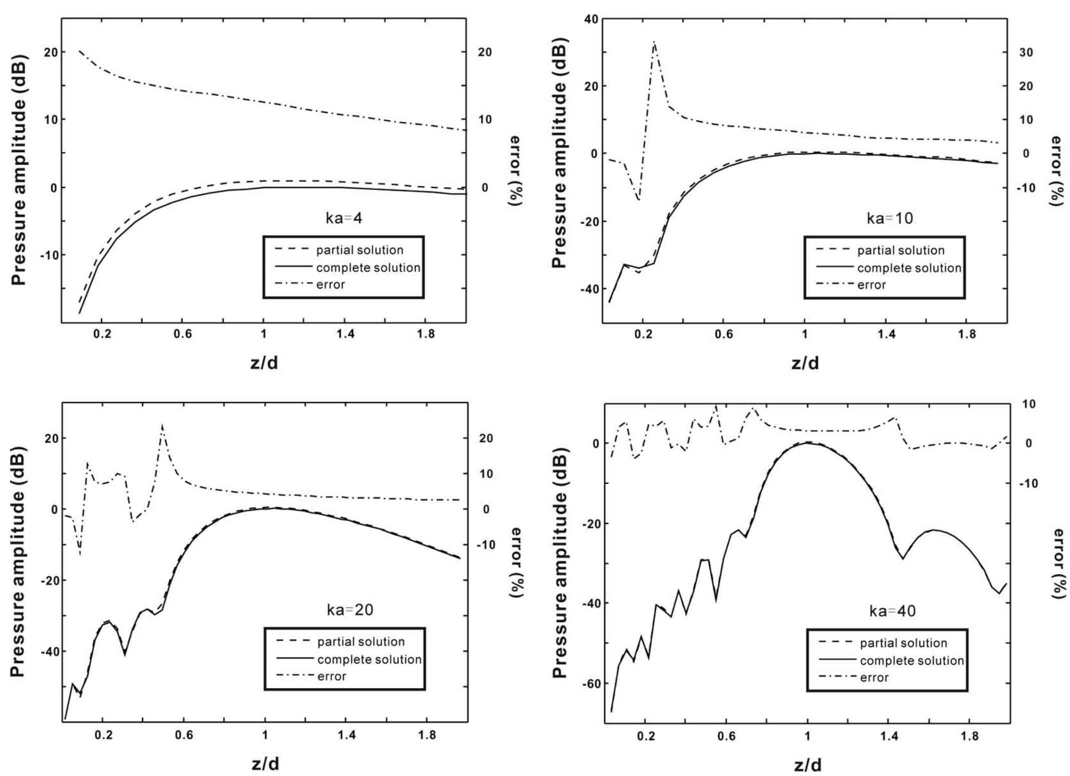


Fig. 5. Comparison of on-axis second harmonics sound pressure distribution between the complete and partial solutions for an aperture angle 80° and a nonlinearity coefficient of 3.5. The values of $ka$ were varied as indicated.

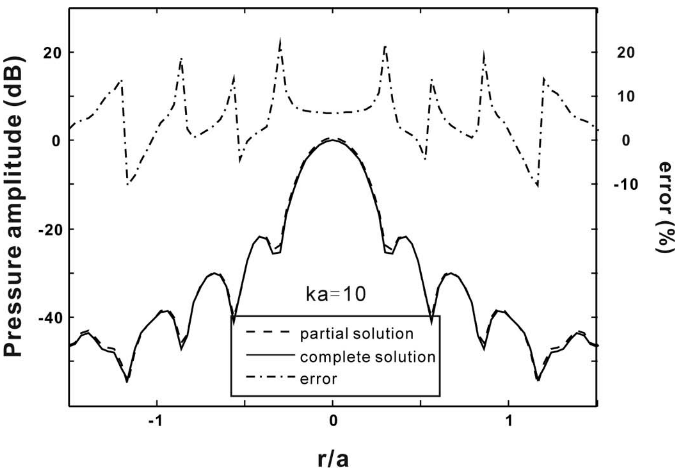


Fig. 6. Comparison of second harmonics sound pressure distribution along the radial direction between the complete and partial solutions at an aperture angle of 80° and a nonlinearity coefficient of 3.5; $ka = 10$.

sound. Four cases were tested, where $ka$ was varied from 4 to 40, the nonlinearity coefficient was 3.5, and the aperture angles were all set at 80°. For the cases in which $ka = 4$ and $ka = 10$, the spatial step-size was chosen to be 1/16 of the wavelength to better represent the tiny transducers. Fig. 5 shows the on-axis second harmonics sound pressure distribution. The least square errors are 0.1169, 0.055, 0.0394, and 0.0326. As expected, the smaller the $ka$ is, the larger the error is, because the plane wave assumption eventually breaks down. It is noted that, the large error area (where error is consistently larger than 15%) is only concentrated in a tiny zone near the source plane, where the source is viewed with very large angles. Therefore, the Westervelt equation might be a good approximation even for small $ka$ as long as the receiver is sufficiently far away from the source. Fig. 6 presents the results along the radial direction for $ka = 10$. The least square error is 0.064. Again, much larger errors have been found compared with the results in Fig. 2.

## IV. Conclusion

This paper investigates the conditions of validity of the Westervelt equation for a focused field in an ideal medium. It has been found that the Westervelt equation provides a very good approximation to the nonlinear acoustic field, even for a strongly focused case. Although it is well known that the Westervelt equation is accurate for quasi-plane wave cases, the result demonstrates this is not necessarily a limit, because negligible error was introduced compared with the full finite-amplitude wave equation. Errors caused by approximations resulting from the equation were observed to increase with decreasing nonlinearity coefficient and $ka$.

Although this study is limited to weakly nonlinear cases, similar behavior is expected for moderately or strongly nonlinear cases, with confirmation planned for future study. Finally, it is noted that there are multiple ways to achieve focusing. Here, phase-based focusing was used, and results may differ from the case of a curved radiator [18], [19], which also demands further study.

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